On the Steiner, geodetic and hull numbers of graphs *

Carmen Hernando[†] Tao Jiang[‡] Mercè Mora[§] Ignacio M. Pelayo[¶]
Carlos Seara[‖]

Abstract

Given a graph G and a subset $W \subseteq V(G)$, a *Steiner W-tree* is a tree of minimum order that contains all of W. Let $S(W)$ denote the set of all vertices in G that lie on some Steiner W-tree; we call $S(W)$ the *Steiner interval* of W. If $S(W) = V(G)$, then we call W a *Steiner set* of G. The minimum order of a Steiner set of G is called the *Steiner number* of G.

Given two vertices u, v in G, a shortest $u - v$ path in G is called a $u - v$ *geodesic*. Let $I[u, v]$ denote the set of all vertices in G lying on some $u - v$ geodesic, and let $J[u, v]$ denote the set of all vertices in G lying on some induced $u - v$ path. Given a set $S \subseteq V(G)$, let $I[S] = \bigcup_{u,v\in S} I[u, v]$, and let $J[S] = \bigcup_{u,v\in S} J[u, v]$. We call $I[S]$ the *geodetic closure* of S and $J[S]$ the *monophonic closure* of S. If $I[S] = V(G)$, then S is called a *geodetic set* of G. If $J[S] = V(G)$, then S is called a *monophonic set* of G. The minimum order of a geodetic set in G is named the *geodetic number* of G.

In this paper, we explore the relationships both between Steiner sets and geodetic sets and between Steiner sets and monophonic sets. We thoroughly study the relationship between the Steiner number and the geodetic number, and address the questions: in a graph G when must every Steiner set also be geodetic and when must every Steiner set also be monophonic. In particular, among others we show that every Steiner set in a connected graph G must also be monophonic, and that every Steiner set in a connected interval graph H must be geodetic.

Keywords: Chordal graph; convexity, geodesic, geodetic set, geodetic number, hull number, monophonic path, monophonic set, Steiner set, Steiner number.

1 Introduction

A *convexity* on a non-empty set V is a family $\mathcal{C}$ of subsets of V (to be regarded as convex sets), such that:

(C1) $\emptyset, V \in \mathcal{C}$.

*Partially supported by projects MCYT-FEDER TIC-2001-2171, MCYT-FEDER BFM 2002-0557, Gen. Cat 2001SGR00224 and by Miami University Faculty Summer Research Grant.

[†]Dpt. Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Spain, carmen.hernando@upc.es

[‡]Dpt. Mathematics and Statistics, Miami University, Oxford, OH, USA, jiangt@muohio.edu

[§]Dpt. Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Spain, merce.mora@upc.es

[¶]*Corresponding author*: Dpt. Matemàtica Aplicada III, Universitat Politècnica de Catalunya, Campus Nord - Edif. C2, Jordi Girona 1-3, 08034 Barcelona, Spain, ignacio.m.pelayo@upc.es

[‖]Dpt. Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Spain, carlos.seara@upc.es

(C2) Arbitrary intersections of convex sets are convex.

(C3) Every nested union of convex sets is convex.

The pair $(V, \mathcal{C})$ is called a *convexity space* (see [13]). The smallest convex set containing a set $A \subseteq V$ is denoted $[A]_{\mathcal{C}}$ and is called the *convex hull* of A. A *graph convexity space* is an ordered pair $(G, \mathcal{C})$ formed by a connected graph $G = (V, E)$, and a convexity $\mathcal{C}$ on V such that $(V, \mathcal{C})$ is a convexity space satisfying the following additional axiom:

(C4) Every member of $\mathcal{C}$ induces a connected subgraph of G.

In what follows, $G = (V, E)$ denotes a connected graph with no loops or multiple edges. The *distance* $d(u, v)$ between two vertices u and v is the length of a shortest $u - v$ path in G. A $u - v$ path ρ is called *monophonic* if it is a chordless path, that is, if $\langle V(\rho)\rangle_G = \rho$. Moreover, the path ρ is called a $u - v$ *geodesic* if it is a shortest $u - v$ path, that is, if $|E(\rho)| = d(u, v)$. The *geodetically closed interval* $I[u, v]$ is the set of vertices of all $u - v$ geodesics. Similarly, the *monophonically closed interval* $J[u, v]$ is the set of vertices of all monophonic $u - v$ paths. For $S \subseteq V$, the *geodetic closure* $I[S]$ of S is the union of all geodesic closed intervals $I[u, v]$ over all pairs $u, v \in S$. The *monophonic closure* is defined similarly. In other words, we have

$$I[S] = \bigcup_{u,v \in S} I[u, v], \quad J[S] = \bigcup_{u,v \in S} J[u, v].$$

The most natural convexities in a graph are *path convexities* defined by a system $\mathcal{P}$ of paths in G. Thus far, two special types of path convexities have received the most attention, the *geodesic convexity* and the *monophonic convexity*. A set $W \subseteq V$ is called *geodetically convex* (or simply *g-convex*) if $I[W] = W$, while it is said to be *geodetic* if $I[W] = V(G)$. Likewise, W is called *monophonically convex* (or simply *m-convex*) if $J[W] = W$, i.e., if W contains all monophonic paths between u and v, for any $u, v \in W$; and it is called *monophonic* if $J[W] = V(G)$.

For a nonempty set W of vertices in a connected graph G, a connected subgraph of G with the minimum number of edges that contains all of W clearly must be a tree; such a tree is called a *Steiner W-tree*. The *Steiner distance* $d_S(W)$ of W is the size of a Steiner W-tree. The *Steiner interval* $S(W)$ of W consists of all vertices that lie on some Steiner W-tree. If $S(W) = V(G)$, then W is called a *Steiner set* for G. The *Steiner number* $st(G)$ of G is defined as the minimum cardinality of a Steiner set of G [2]. The *Steiner set problem* for G is concerned with Steiner sets in G and the Steiner number $st(G)$ of G.

In [2], it was shown that every Steiner set in a graph G is also geodetic (Theorem 3.2). Unfortunately, this particular result turned out to be wrong and was disproved by Pelayo [12]. This, however, raises the natural questions: (1) under what conditions must every Steiner set also be geodetic and (2) whether there are any general relationships between Steiner sets and geodetic sets in a graph G. We address those questions in this paper. Along the way, we also consider the relationship between Steiner sets and monophonic sets. In particular, we show that there is no general relationship between Steiner sets and geodetic sets in a graph G; this is illustrated by showing that there is no direct relationship between the Steiner number $st(G)$ and the geodetic number $gn(G)$. On the other hand, we show that in every connected graph G, indeed every Steiner set must be monophonic, and that if G is a connected interval graph then every Steiner set in G must be geodetic.

The rest of the paper is organized as follows. In Section 2, we consider the relationship between Steiner sets and geodesic sets. In Section 3, we establish the fact that every Steiner set in a connected graph G must be monophonic. Finally, in Section 4 we consider the relationship between Steiner sets and geodetic sets in the family of chordal graphs. Among others, we show that in a connected interval graph G, every Steiner set is geodetic.

We close the introduction by posing the following problem which we partially addressed in this paper but deserves further study.

Problem. *Characterize graphs G for which every Steiner set in G is also geodetic.*

2 Geodetic approach

Throughout this section, we only consider the geodesic convexity for a connected graph $G = (V, E)$. A subset S of vertices of G is said to be a *hull set* if its (geodesic) convex hull $[S]_g$ covers all the graph, i.e., if $[S]_g = V$. Moreover, S is called *geodetic* if $I[S] = V$. The *hull number* $hn(G)$ of a graph G is defined as the minimum cardinality of a hull set. The *geodetic number* $gn(G)$ of G is the minimum cardinality of a geodetic set. Certainly, $hn(G) \leq gn(G)$.

Although it has been shown that determining the geodetic number (resp. the hull number) of a graph is an NP-hard problem [8], it is rather simple to obtain these two parameters for a wide range of classes of graphs as paths, cycles, trees, (bipartite) complete graphs, wheels and hypercubes (see Table 1).

In [2], the authors stated that every Steiner set is geodetic (Theorem 3.2). This result was disproved by Pelayo [12] by counterexample. For the sake of completeness, we include the so-called graph J_7, which contains a Steiner set that is not geodetic (Figure 1). An immediate corollary of the previous wrong theorem is that $gn(G) \leq st(G)$. This result was also proved to be false in [12]. Consider, for instance, the graph $J_7 + ab$ (see Figure 1). One can quickly check that the set $\{u, v, w\}$ is a minimum Steiner set. Hence, $st(J_7 + ab) = 3$. On the other hand, no subset of $V(J_7 + ab)$ of cardinality 3 or less is geodetic. Hence, $st(J_7 + ab) < gn(J_7 + ab)$.

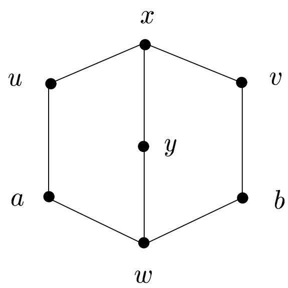

Figure 1: Graph J_7. Notice that the set $\{u, v, w\}$ is both a Steiner and a hull set but it is not geodetic, meanwhile the set $\{a, y, v\}$ is both a Steiner and a geodetic set.

At this point, it seems appropriate to ask the following question: *Is every Steiner set a hull set*? As in the previous cases, this statement is also false, and here is a counterexample.

Consider the graph M_{13} showed in Figure 2a. Certainly, $W = \{1, 4, 7\}$ is a minimum Steiner set, and it is neither geodetic nor a hull set, since $[W]_g = I[W] = V(M_{13}) - \{a, b, c, d\}$. Observe that the equality $[W]_g = I[W]$ is a direct consequence of the fact that the 9-cycle induced by $V(M_{13}) - \{a, b, c, d\}$ is convex.

Before continuing, let us introduce two basic definitions and two lemmas. A *half-space* is a convex set, of which the complement is convex as well. For example, the set $\{a, b, c, d\}$ of the graph M_{13} showed in Figure 2a is a half-space. A vertex v in a graph G is an *extreme vertex* if the subgraph induced by its neighborhood $N(v)$ is a clique, in other words, if $V-v$ is convex. The set of all extreme vertices of G is denoted by $Ext(G)$.

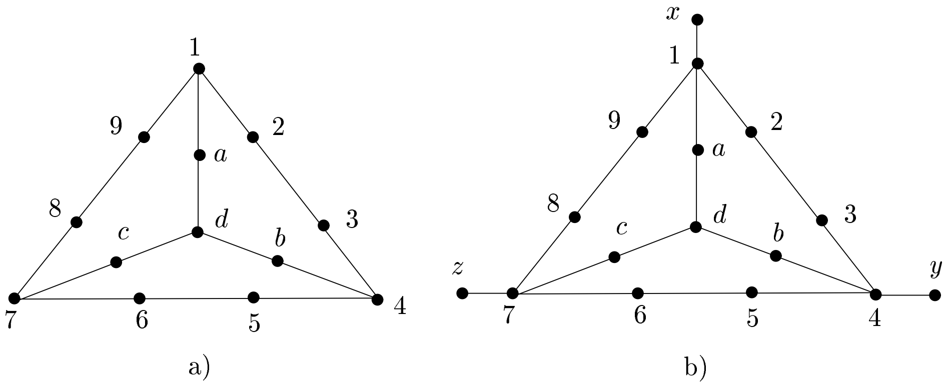

Figure 2: a) Graph M_{13} satisfying $hn(M_{13}) = st(M_{13}) = 3 < 4 = gn(M_{13})$, b) graph M_{16}.

Lemma 2.1. *Let $G = (V, E)$ be a connected graph and let $\Omega \subsetneq V$ be a nonempty half-space. If $W \subseteq V$ is a hull set, then $W \cap \Omega \neq \emptyset$, $W \cap (V \setminus \Omega) \neq \emptyset$.*

Proof. Suppose, for example, that $W \subseteq \Omega$. Then, $[W]_g \subseteq \Omega$, since $[W]_g$ is the smallest convex set containing W. Hence, W is not a hull set. □

Lemma 2.2. *Let $G = (V, E)$ be a connected graph. If $W \subseteq V$ is a hull set (resp. a geodetic set, a Steiner set), then $Ext(G) \subseteq W$.*

Proof. Assume that $v \in Ext(G)$ such that $v \notin W$. This means that $[W]_g \subseteq V - v$, since the set $V - v$ is convex. Hence, W is neither a hull set nor a geodetic set. Suppose that W is a Steiner set of G. This means that there exists a Steiner W-tree T such that $v \in V(T)$. Certainly, $deg_T(v) = r \geq 2$, since every leaf of T is in W. If $N_T(v) = \{a_1, \ldots, a_r\}$, then for every $i \in \{1, \ldots, r-1\}$, $a_i a_{i+1} \in E$, since $N(v)$ is a clique of G. In consequence, the subgraph $T' = (T - v) + \{a_1 a_2, \ldots, a_{r-1} a_r\}$ is a tree satisfying both $W \subseteq V(T')$ and $|V(T')| < |V(T)|$, contradicting the fact that T is a Steiner W-tree. □

Returning to the graph M_{13} (Figure 2a), it is easy to see that the set $S = \{3, 8, d\}$ satisfies $I[S] = V \setminus \{a, 5, 6\}$, $[S]_g = I^2[S] = V$. Moreover, observe that for every $u, v \in V(M_{13})$, there exists a unique $u - v$ geodesic. Hence, $hn(M_{13}) = 3$. Nevertheless, the inequality $hn(G) \leq st(G)$ is not true in general. To prove this statement, consider the so-called graph M_{16} obtained from M_{13} adding the set of vertices $\Lambda = \{x, y, z\}$ as it is shown in Figure 2b. Certainly, $Ext(M_{16}) = \Lambda$. Thus, by Lemma 2.2, we obtain that

$st(M_{16}) = 3$ and $hn(M_{16}) = 4$, since Λ is a Steiner set, $[\Lambda]_g = I[\Lambda] = V(M_{16}) - \{a, b, c, d\}$, and $\{x, y, z, d\}$ is a hull set.

G	P_n	C_{2l}	C_{2l+1}	T_n	K_n	$K_{p,q}$ $(2 \leq p \leq q)$	$W_{1,p}$ $(p \geq 4)$	Q_n
$hn(G)$	2	2	3	$\sharp$ leaves	n	2	$\lceil \frac{p}{2} \rceil$	2
$mn(G)$	2	2	3	$\sharp$ leaves	n	$\min\{4, p\}$	2	2
$gn(G)$	2	2	3	$\sharp$ leaves	n	$\min\{4, p\}$	$\lceil \frac{p}{2} \rceil$	2
$st(G)$	2	2	3	$\sharp$ leaves	n	p	$p - 2$	2

Table 1: Geodetic, monophonic, hull and Steiner number of some classes of graphs.

After this collection of counterexamples, what remains to be done is to ask the following question: *Is there any other general relationship among the parameters $hn(G)$, $gn(G)$ and $st(G)$, apart from the known inequality: $hn(G) \leq gn(G)$?* The following results show that unless we restrict ourselves to a specific class of graphs, the answer is negative.

2.1 Some partial and particular results

In this subsection, we evaluate the parameters $hn(G)$, $gn(G)$ and $st(G)$ for some particular connected graphs which will allow us to obtain a number of partial realization results, each of them involving two of the three mentioned parameters.

Lemma 2.3. *The graph $J_{7,m}$ (see Figure 3) satisfies $hn(J_{7,m}) = 2$, $gn(J_{7,m}) = m + 2$.*

Proof. Consider the graph $J_{7,m}$ obtained from J_7 (see Figure 1) by blowing up the 2-path $x - y$ into m pieces as it is showed in Figure 3. Clearly, the set $A = \{a, v\}$ satisfies $I[A] = V(J_{7,m}) \setminus \{y_i\}_{i=1}^{m}$, $[A]_g = I^2[A] = V(J_{7,m})$. Hence, $hn(J_{7,m}) = 2$. Observe that for every $i \in \{1, 2, \ldots, m\}$ and for every $B \subseteq V(J_{7,m}) \setminus \{x_i, y_i\}$, $I[B] \subsetneq V(J_{7,m})$. From this fact, it is easy to see that $C = \{a, v\} \cup \{y_i\}_{i=1}^{m}$ is a minimum geodetic set. This means that $gn(J_{7,m}) = m + 2$. □

Proposition 2.1. *For every pair α, β of integers, $2 \leq \alpha \leq \beta$, there exists a connected graph G such that $hn(G) = \alpha$, $gn(G) = \beta$.*

Proof. Although this statement was already proved in [1], we include this simpler proof. For $\alpha = \beta$, K_α has the desired properties. Suppose that $\alpha < \beta$ and consider the so-called graph $J_{7,m,l}$ obtained from $J_{7,m}$ by adding a set of l vertices (leaves) $\{a_1, \ldots, a_l\}$ (Figure 4a). As a direct consequence from Lemmas 2.2 and 2.3, we conclude that $\{v\} \cup \{a_j\}_{j=1}^{l}$ (resp. $\{v\} \cup \{a_j\}_{j=1}^{l} \cup \{y_i\}_{i=1}^{m}$) is a minimum hull (resp. geodetic) set. Thus, taking $m = \beta - \alpha$ and $l = \alpha - 1$, we have a graph G satisfying $hn(G) = \alpha$, $gn(G) = \beta$. □

Proposition 2.2. *For every pair of integers $\alpha, \beta \geq 3$, there exists a connected graph G such that $st(G) = \alpha$, $gn(G) = \beta$.*

Proof. The case $\alpha \geq \beta$ was proved in [2]. Assume thus that $\alpha < \beta$. Consider the graph $J_{7,m}$ (Figure 3). By Lemma 2.3 we know that $C = \{a, v\} \cup \{y_i\}_{i=1}^{m}$ is a minimum geodetic set. Certainly, $\{u, v, w\}$ is a minimum Steiner set. Next, consider the so-called graph

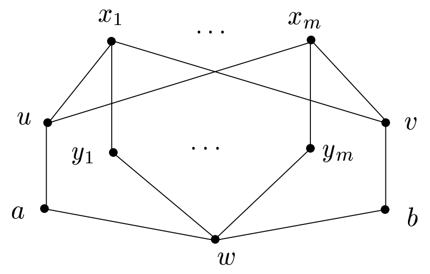

Figure 3: Graph $J_{7,m}$.

$J^k_{7,m}$ obtained from $J_{7,m}$ by adding a set $\{v_1, \ldots, v_k\}$ of k leaves (Figure 4b). As a direct consequence from Lemma 2.2, we conclude that $\{u, w\} \cup \{v_j\}_{j=1}^k$ (resp. $\{a\} \cup \{v_j\}_{j=1}^k \cup \{y_i\}_{i=1}^m$) is a minimum Steiner (resp. geodetic) set. Hence, taking $m = \beta - \alpha + 1$ and $k = \alpha - 2$, we have a graph G satisfying $st(G) = \alpha$, $gn(G) = \beta$. $\square$

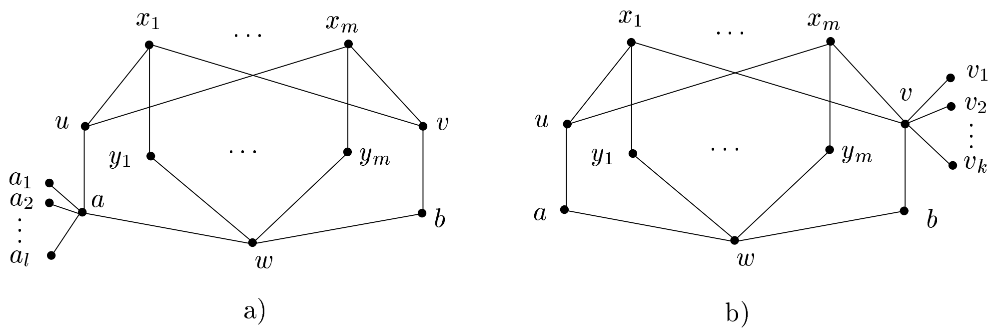

Figure 4: a) Graph $J_{7,m,l}$, b) graph $J^k_{7,m}$.

Lemma 2.4. *The graph B_{lm} (see Figure 5) satisfies $hn(B_{lm}) = m + 1$, $st(B_{lm}) = m + l$.*

Proof. Certainly, the set of vertices $\{a_1\} \cup \{b_i\}_{i=1}^m$ is a minimum hull set of the graph B_{lm}, since $Ext(B_{lm}) = \{b_i\}_{i=1}^m$. It is also easy to see that $\{a_i\}_{j=1}^l \cup \{b_i\}_{i=1}^m$ is a minimum Steiner set. As a consequence, we obtain that $hn(B_{lm}) = m + 1$ and $st(B_{lm}) = l + m$. $\square$

Lemma 2.5. *The graph M_{13m} (see Figure 6) satisfies $st(M_{13m}) = 3$, $hn(M_{13m}) = m + 2$ and $gn(M_{13m}) = m + 3$.*

Proof. Consider the so-called graph M_{13m} obtained from M_{13} (Figure 2a) by blowing up the subgraph H of M_{13} induced by the set $\Omega = \{a, b, c, d\}$ (i.e., $H = \langle\Omega\rangle_{M_{13}}$), into m pieces $\{H_1, H_2, \ldots, H_m\}$ (Figure 6). That is:

$$M_{13m} = C_9 \cup (\bigcup_{i=1}^{m} A_i).$$

To be more precise:

$$V(M_{13m}) = V(M_{13} - \Omega) \cup (\cup_{i=1}^m V(H_i)),$$

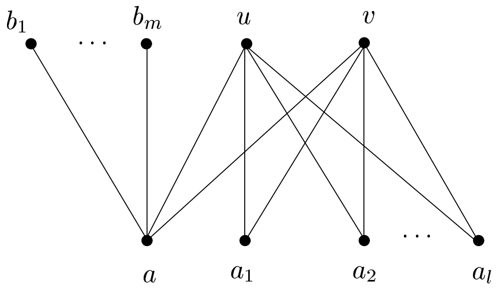

Figure 5: The graph B_{lm}.

$$E(M_{13m}) = E(M_{13} - \Omega) \cup (\cup_{i=1}^{m} E(H_i)) \cup (\cup_{i=1}^{m} 1a_i) \cup (\cup_{i=1}^{m} 4b_i) \cup (\cup_{i=1}^{m} 7c_i).$$

Clearly, $\{1, 4, 7\}$ is a minimum Steiner set of M_{13m}. Moreover, observe that, for every $i \in \{1, 2, \ldots, m\}$, H_i is a half-space of M_{13m}. Hence, from Lemma 2.1, we easily derive that $\{1, 4, 7\} \cup \{d_i\}_{i=1}^{m}$ is a minimum geodetic set and $\{3, 8\} \cup \{d_i\}_{i=1}^{m}$ is a minimum hull set. □

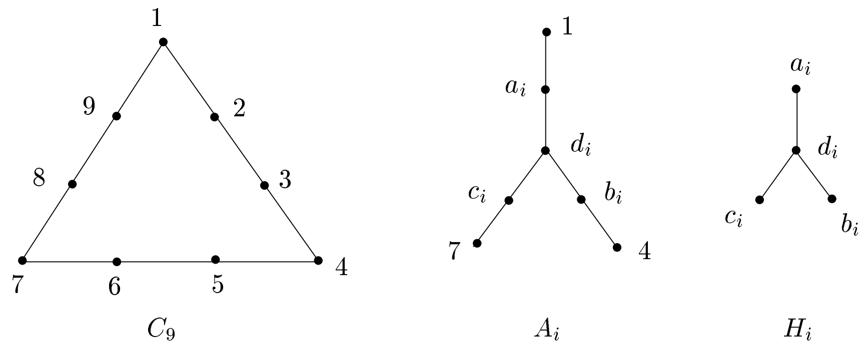

Figure 6: Components of the graph M_{13m}.

Proposition 2.3. *For every pair of integers $\alpha, \beta \geq 3$, there exists a connected graph G such that $st(G) = \alpha$, $hn(G) = \beta$.*

Proof. First, suppose that $\beta \leq \alpha$. Consider the bipartite graph B_{lm} showed in Figure 5. Using Lemma 2.4 and taking $m = \beta - 1$, $l = \alpha - \beta + 1$ we are done.

Second, assume that $\alpha \leq \beta$. Consider the so-called graph M_{13ml}, obtained from M_{13m} (Figure 6) by adding a set of l vertices $\{x_1, x_2, \ldots, x_l\}$ so that:

$$V(M_{13ml}) = V(M_{13m}) \cup \{x_i\}_{i=1}^{l}, \qquad E(M_{13ml}) = E(M_{13m}) \cup \{7x_i\}_{i=1}^{l}.$$

Applying Lemmas 2.2 and 2.5, we obtain that $\{1, 4\} \cup \{x_i\}_{i=1}^{l}$ (resp. $\{1, 4\} \cup \{d_i\}_{i=1}^{m} \cup \{x_i\}_{i=1}^{l}$) is a minimum Steiner (resp. hull) set. Thus, taking $l = \alpha - 2$ and $m = \beta - \alpha$, we have a graph G verifying $st(G) = \alpha$, $hn(G) = \beta$. □

2.2 The realization theorem

In this final subsection, we present our main realization result, involving all of three parameters $hn(G)$, $gn(G)$ and $st(G)$. To begin with, we need to show three technical lemmas.

Lemma 2.6. *Consider the graph G of Figure 7, with $k \geq 2$, $p \geq 0$, $m \geq 0$. Then,*

1. $W_1 = \{x\} \cup \{v_i\}_{i=1}^k$ *is a minimum hull set.*
2. $W_2 = W_1 \cup \{w_i\}_{i=1}^p$ *is a minimum geodetic set.*
3. $W_3 = W_2 \cup \{u_i\}_{i=1}^m$ *is a minimum Steiner set.*

Proof.

1. Certainly, the vertex set W_1 is a hull set, since $I[W_1] = V(G) \setminus \{w_i\}_{i=1}^p$, $I^2[W_1] = V(G)$. On the other hand, observe that $Ext(G) = W_1$. Hence, from Lemma 2.2, we immediately derive that W_1 is a hull set of minimum cardinality.

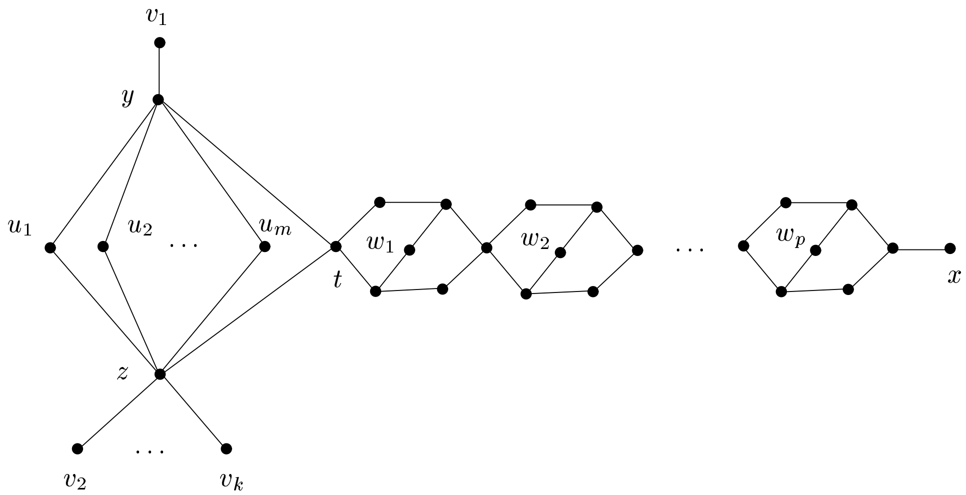

Figure 7: $3 \leq hn(G) \leq gn(G) \leq st(G)$.

2. The vertex set W_2 is geodetic, since $I[W_2] = V(G)$. Notice that, for every $j \in \{1, \dots, p\}$, if W is a vertex set such that $W \cap (w_j \cup N(w_j)) = \emptyset$, then $w_j \notin I[W]$. Moreover, we know that a geodetic set must contain $Ext(G) = W_1$ and $I[W_1] = V(G) \setminus \{w_i\}_{i=1}^k$. In consequence, we can conclude that W_2 is a minimum geodetic set.

3. Let W be a vertex set of the graph G. Observe that, for every $j \in \{1, \dots, p\}$, if W satisfies $W \cap (w_j \cup N(w_j)) = \emptyset$, then $w_j \notin S(W)$. Next, assume that W is a Steiner set. Hence, W must contain $\{w_j\}_{j=1}^k$ and, by Lemma 2.2, we know that $Ext(G) = W_1 \subseteq W$. Moreover, notice that, for every Steiner W-tree, the set $\{y, z, t\}$ must be contained in $V(T)$, since its elements are all of them cut vertices of G. Since $N(u_i) = \{y, z\}$, if $u_i \notin W$, then u_i does not belong to any Steiner W-tree. Hence, we have proved that, for every Steiner set W, $\{u_i\}_{i=1}^m \subset W$. All of these facts allow

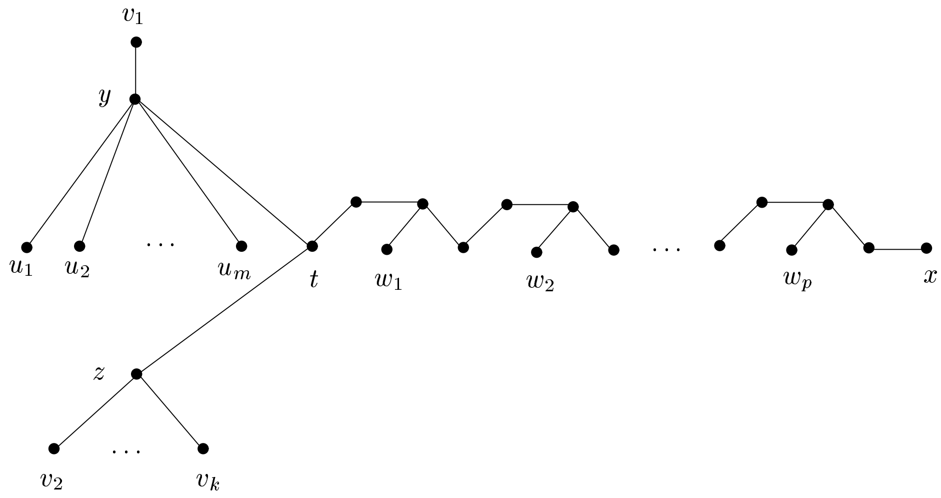

Figure 8: Steiner tree of the Steiner set $W_3 = \{x\} \cup \{v_i\}_{i=1}^k \cup \{w_i\}_{i=1}^p \cup \{u_i\}_{i=1}^m$.

us to conclude that W_3 is a minimum Steiner set of G, since $S(W_3) = V(G)$ (see Figure 8).

$\square$

Lemma 2.7. *Consider the graph G of Figure 9, with $m \geq 1$, $k \geq 0$, $p \geq 1$. Then,*

1. *$W_1 = \{x, y\} \cup \{v_i\}_{i=1}^m$ is a minimum hull set.*
2. *$W_2 = W_1 \cup \{u_i\}_{i=1}^k$ is a minimum Steiner set.*
3. *$W_3 = W_2 \cup \{w_i\}_{i=1}^p$ is a minimum geodetic set.*

Proof. 1. Certainly, the vertex set W_1 is a hull set, since $I[W_1] = V(G) \setminus \{u_i\}_{i=1}^k \setminus \{w_i\}_{i=1}^p$, $[W_1]_g = I^2[W_1] = V(G)$. On the other hand, observe that $Ext(G) = \{x\} \cup \{v_i\}_{i=1}^m$ and $y \notin [Ext(G)]_g$. Hence, Lemma 2.2 allows us to conclude that W_1 is a hull set of minimum cardinality.

2. Let W be a vertex set of the graph G. Observe that, for every $j \in \{1, \ldots, k\}$, if W satisfies $W \cap (u_j \cup N(u_j)) = \emptyset$, then $u_j \notin S(W)$. Notice also that if $W \cap (y \cup N(y)) = \emptyset$, then $y \notin S(W)$. Hence, W_2 is a minimum Steiner set since $Ext(G) = \{x\} \cup \{v_i\}_{i=1}^m$ and $S(W_2) = V(G)$.

3. Let W be a vertex set of the graph G. Observe that,

   (a) for every $j \in \{1, \ldots, k\}$, if W satisfies $W \cap (u_j \cup N(u_j)) = \emptyset$, then $u_j \notin I[W]$;

   (b) for every $i \in \{1, 2, \ldots, p\}$, if $W \subseteq V(G) \setminus \{c_i, w_i\}$, then $w_i \notin I[W]$;

   (c) if $\{a, y, b\} \subseteq V(G) \setminus W$, then $I[W] \subset V(G) \setminus \{a, b\}$.

   From this facts, it is easy to conclude that W_3 is a minimum geodetic set of G, since $I[W_3] = V(G)$.

$\square$

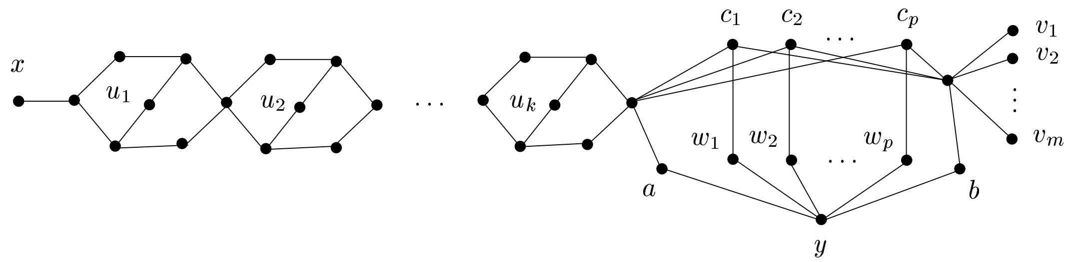

Figure 9: $3 \leq hn(G) \leq st(G) < gn(G)$.

Finally, consider the graph

$$G = T_k \cup \left(\bigcup_{i=1}^{m} A_i \right) \cup \left(\bigcup_{j=1}^{p} B_j \right),$$

constructed with the graphs illustrated in Figure 10. For example, the graph M_{16} showed in Figure 2b), is precisely the graph G for the values $k = 1$, $m = 1$ and $p = 0$.

Lemma 2.8. *Consider the graph $G = T_k \cup (\cup_{i=1}^{m} A_i) \cup \left(\cup_{j=1}^{p} B_j\right)$ just defined. Then,*

1. *$W_1 = \{x, y\} \cup \{z_i\}_{i=1}^{k}$ is a minimum Steiner set.*
2. *$W_2 = W_1 \cup \{d_i\}_{i=1}^{m}$ is a minimum hull set.*
3. *$W_3 = W_2 \cup \{f_i\}_{i=1}^{p}$ is a minimum geodetic set.*

Proof.

1. Clearly, W_1 is a Steiner set since every vertex of G belongs to some Steiner W_1-tree (observe that the order of any Steiner W_1-tree is $k+9$). Moreover, from Lemma 2.2, we derive that W_1 is a minimum Steiner set since $Ext(G) = W_1$.
2. Certainly, the vertex set W_2 is a hull set, since $I[W_2] = V(G) \setminus \{g_j, h_j\}_{j=1}^{p}$, $I^2[W_2] = V(G)$ and $[W_2]_g = I^2[W_2] = V(G)$. On the other hand, observe that $Ext(G) = \{x, y\} \cup \{z_i\}_{i=1}^{k} = W_1$ and notice also that, for every $i \in \{1, \ldots, m\}$, the vertex set $\{a_i, b_i, c_i, d_i\}$ is a half-space. Hence, Lemmas 2.1 and 2.2 allow us to conclude that W_2 is a hull set of minimum cardinality.
3. The vertex set W_3 is geodetic, since $I[W_3] = V(G)$. Observe that, for every $j \in \{1, 2, \ldots, p\}$, if $W \subseteq V(G) \setminus \{e_j, f_j, g_j, h_j\}$, then $I[W] \subsetneq V(G)$. From this fact, it is easy to conclude that W_3 is a minimum geodetic set of G.

□

Theorem 2.1. *For every triple a, b, c of integers with $3 \leq a \leq b \leq c$, there exists a connected graph G such that:*

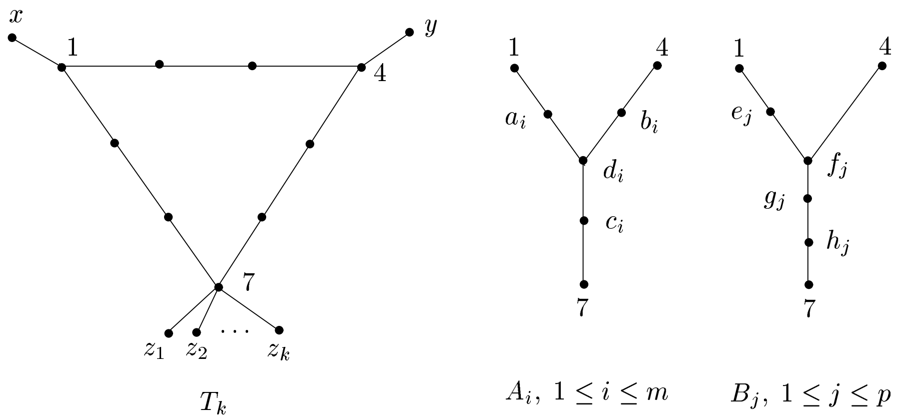

Figure 10: $3 \leq st(G) \leq hn(G) \leq gn(G)$.

1. $hn(G) = a$, $gn(G) = b$, $st(G) = c$

2. $hn(G) = a$, $st(G) = b$, $gn(G) = c$

3. $st(G) = a$, $hn(G) = b$, $gn(G) = c$

Proof.

Case 1: According to Lemma 2.6, the graph in Figure 7 satisfies the equalities $hn(G) = k+1$, $gn(G) = k+1+p$, $st(G) = k+1+p+m$, for $k \geq 2$, $p \geq 0$ and $m \geq 0$. Hence, if we take $k = a-1$, $p = b-a$ and $m = c-b$ we are done.

Case 2: For $m \geq 1$ and $k \geq 0$, the graph in Figure 11 satisfies $hn(G) = m+1$, $gn(G) = m+1+k$, $st(G) = m+1+k$. Concretely, the set $W_1 = \{x\} \cup \{v_i\}_{i=1}^{m}$ is a minimum hull set and $W_2 = W_1 \cup \{u_i\}_{i=1}^{k}$ is both a minimum geodetic and a minimum Steiner set. As a result, if we take $m = a-1$ and $k = b-a$ we obtain a graph G satisfying $hn(G) = a \leq b = st(G) = gn(G)$.

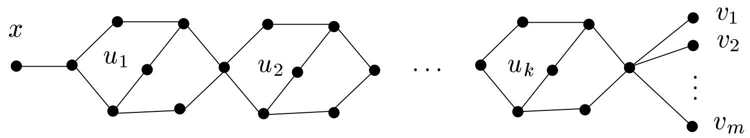

Figure 11: $3 \leq hn(G) \leq st(G) = gn(G)$.

Finally, suppose $a \leq b < c$. The graph in Figure 9 satisfies the equalities $hn(G) = m+2$, $st(G) = m+2+k$, $gn(G) = m+2+k+p$, for $m \geq 1$, $k \geq 0$ and $p \geq 1$ (see Lemma 2.7). If we take $m = a-2$, $k = b-a$ and $p = c-b$ we obtain a graph G such that $hn(G) = a \leq b = st(G) < gn(G) = c$.

Case 3: The graph G evaluated in Lemma 2.8 satisfies the equalities $st(G) = k+2$, $hn(G) = k+2+m$ and $gn(G) = k+2+m+p$, for $k \geq 1$, $m \geq 0$ and $p \geq 0$. If we take $k = a-2$, $m = b-a$, $p = c-b$ we obtain a graph G such that $3 \leq a = st(G) \leq hn(G) = b \leq c = gn(G)$.

□

3 Monophonic approach

Throughout this section, we only consider the monophonic convexity (see [5]). A subset of vertices S of a connected graph $G = (V, E)$ is a said to be a (monophonic) *hull set* if its (monophonic) convex hull $[S]_m$ covers the graph, i.e., if $[S]_m = V$. Moreover, S is called *monophonic* if $J[S] = V$. The *monophonic hull number* $mhn(G)$ of G is the minimum cardinality of a (monophonic) hull set. The *monophonic number* $mn(G)$ of G is the minimum cardinality of a monophonic set. Certainly, $mhn(G) \leq mn(G) \leq gn(G)$, since every geodetic set is monophonic and every monophonic set is a monophonic hull set. Nevertheless, it is not true that every (geodesic) hull set be monophonic. For example, the set $W = \{a_1\} \cup \{b_i\}_{i=1}^{m}$ of the graph B_{lm} illustrated in Figure 5 is a hull set satisfying $J[W] = I[W] = V(B_{lm}) \setminus \{a_i\}_{i=2}^{l}$.

We have seen that not every Steiner set is a (geodesic) hull set, it is then natural to ask then *Is every Steiner set monophonic*? Very pleasantly, this time, the answer turns out to be affirmative.

Lemma 3.1. *Let $G = (V, E)$ be a connected graph. Let $W \subseteq V$, and let T be a Steiner W-tree in G. Then $V(T) \subseteq J[W]$.*

Proof. Let H denote the subgraph of G induced by $V(T)$, i.e., $H = \langle V(T) \rangle$. For each pair $u, v \in W$, let $\rho_{u,v}$ denote an induced u, v-path in H. Note that since H is an induced subgraph of G, $\rho_{u,v}$ is also an induced path in G. In particular, we have $V(\rho_{u,v}) \subseteq J[u, v]$. Let $F = \bigcup_{u,v \in W} \rho_{u,v}$, i.e. F is the union of $\rho_{u,v}$ over all pairs $u, v \in W$. By our discussion above, we have $V(F) = \bigcup_{u,v \in W} V(\rho_{u,v}) \subseteq J[W]$.

Clearly, F is a connected subgraph of H (and hence a connected subgraph of G) that contains all of W. Let F' denote a spanning tree of F, then F' is a subtree in G that contains all of W such that $V(F') \subseteq V(T)$. Since T is a Steiner W-tree, we see that F' must also be a Steiner W-tree and $V(F') = V(T)$. Consequently, we have $V(T) = V(F') = V(F) \subseteq J[W]$. □

Theorem 3.1. *Every Steiner set of a connected graph $G = (V, E)$ is monophonic.*

Proof. Let $W \subseteq V$ be a Steiner set of G. Then $V(G)$ is the set of all vertices that lie in some Steiner W-tree. By Lemma 3.1, for each Steiner W-tree T in G we have $V(T) \subseteq J[W]$. Hence, we have $V(G) \subseteq J[W]$. This shows that W is a monophonic set in G. □

As an immediate consequence of Theorem 3.1, we have

Corollary 3.1. *For every connected graph $G = (V, E)$, $mhn(G) \leq mn(G) \leq st(G)$.*

A *distance-hereditary graph* is a graph in which every monophonic path is a geodesic [9]. As a consequence, Theorem 3.1 allows us to derive the following corollary.

Corollary 3.2. *In any distance-hereditary graph, every Steiner set is geodetic.*

Observe that a key ingredient in our proofs above is the fact that every induced path in an induced subgraph H of G is also an induced path in G. Hence, $J_H[W] \subseteq J_G[W] = J[W]$.

This however does not transfer to shortest paths. In other words, a shortest path in H needs not be a shortest path in G. In general, $I_H[W] \subseteq I_G[W]$ does not hold! This is a key difference between geodetic closures and monophonic closures, and it may help explain why every Steiner set is monophonic but needs not be geodetic.

4 The Steiner set problem in chordal graphs

We have seen that in a connected graph G, not every Steiner set needs to be geodetic. That being the case, it is reasonable to then ask under what conditions must every Steiner set be geodetic as well. We approach this question in this section by considering a special class of graphs G, namely chordal graphs.

A *chordal graph* is a graph containing no induced cycle of length at least 4. Chordal graphs form an important subclass of perfect graphs, and have been extensively studied in different ways, including within the context of convexity in graphs (see [3, 5, 6] for more details).

First, consider the chordal graph illustrated in Figure 12a. Observe that the vertex set $S = \{u, v, w\}$ is a Steiner set, but it is not geodetic since $I[S] = V(T_7) - y$. Hence, even for the class of chordal graphs the claim: *Every Steiner set is geodetic*, is not true.

Let us now shift our attention to some important subclasses of chordal graphs including complete graphs, trees, Ptolemaic graphs, split graphs, interval graphs and strongly chordal graphs.

Certainly, there is a unique Steiner set in every complete graph, namely, the set of all its vertices, which is also its unique geodetic set. It is also clear that for every tree, a vertex set is a Steiner set if and only if it is geodetic. A *Ptolemaic graph* is a chordal graph which contain no 3-fan as induced subgraph (see Figure 12b). Edward Howorka proved in [10] that a graph is Ptolemaic if and only if it is both chordal and distance-hereditary. Hence, from Corollary 3.2 we immediately derive that, for the class of Ptolemaic graphs the claim: *Every Steiner set is geodetic* is true.

Another interesting and large subclass of chordal graphs is the family of split graphs. A *split graph* is a chordal graph with a chordal complement; this terminology arises because a graph G is a split graph if and only if there is a partition $V(G) = V_1 \cup V_2$ where V_1 is an independent set and V_2 is a clique (see [7]). Observe that the chordal graph showed in Figure 12a is also a split graph. In consequence, for the class of split graphs the claim: *Every Steiner set is geodetic* is not true.

Definition 4.1. *A graph G is an interval graph if there exists a one-to-one mapping I from $V(G)$ to the set of closed intervals on the real line such that two vertices x, y are adjacent in G if and only if the intervals $I(x)$ and $I(y)$ intersect. The mapping I is called an interval representation of G.*

Interval graphs form an important subfamily of chordal graphs and have been extensively studied in the literature. C. G. Lekkerkerker and J. Ch. Boland proved in [11] that interval graphs are precisely those chordal graphs without induced asteroidal triples, i.e., independent sets of three vertices such that each pair is joined by a path that avoids the neighborhood of the third (see Figure 12c). We will now see that for an interval graph G, the claim: *Every Steiner set is geodetic* is indeed true!

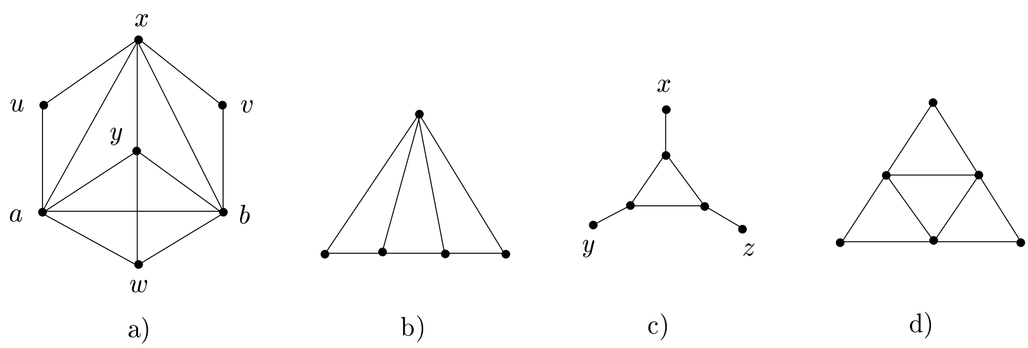

Figure 12: a) Graph T_7, b) 3-fan, c) $\{x, y, z\}$ is an asteroidal triple, d) 3-sun.

Theorem 4.1. *Let G be an interval graph. Let W be a Steiner set of G. Then W is also a geodetic set of G.*

Proof. Let I be an interval representation of G. For each closed interval J in the real line, let $a(J)$ and $b(J)$ denote the left endpoint and the right endpoint of J, respectively.

Let x be a vertex in W such that $I(x)$ has the minimum left endpoint, i.e., $x \in W$ is chosen such that $a(I(x)) = \min\{a(I(w)) : w \in W\}$. Let y be a vertex in W such that $I(y)$ has the maximum right endpoint, i.e., $b(I(y)) = \max\{b(I(w) : w \in W\}$. Suppose first that $x = y$. Then $I(x)$ contains $I(w)$ for all $w \in W$. This means that x is adjacent to all $W - x$, in which case every Steiner tree of W contains only W, implying that $V(G) = W$ and there is nothing to prove.

So we may assume that $x \neq y$. Given any vertex $z \in V(G) - W$, we prove that there exist $w', w'' \in W$ such that z lie on some shortest $w' - w''$ path in G. Since $S(W) = V(G)$ there exists a Steiner tree T of W that contains z. Let P denote the unique path in T between x and y. Let $a^* = a(I(x))$ and $b^* = b(I(y))$. Let $I(P) = \bigcup_{u \in V(P)} I(u)$. Note first that $[a^*, b^*] \subseteq I(P)$. By our choice of x, y, if $v \in W$, then $a(I(v)) \geq a^*$ and $b(I(v)) \leq b^*$. Hence $I(v) \subseteq [a^*, b^*] \subseteq I(P)$. In particular, there exists some vertex u on P such that $I(v) \cap I(u) \neq \emptyset$. Thus, if v is not already on P then it is adjacent to some vertex on P. Let L denote the set of vertices of W not in P. Consider the tree T' obtained from P by adding those vertices of L as leaves to appropriate vertices on P. We have $V(T') = V(P) \cup L$, and $W \subseteq V(T') \subseteq V(T)$. Since T is a Steiner tree of W, we must have $V(T') = V(T)$. Thus, T' is also a Steiner tree of W that contains z. Furthermore, the structure of T' implies that z lies on P.

Now, let w', w'' be vertices of W on P such that the portion of P between w' and w'', denoted $P[w', w'']$, contains z and $V(P[w', w'']) \cap W = \{w', w''\}$; such w', w'' clearly exist. We claim that $P[w', w'']$ is a shortest $w' - w''$ path in G, which will complete our proof. Suppose otherwise that there exists a $w' - w''$ path P' in G shorter than $P[w', w'']$. Consider $P'' = P - P[w', w''] \cup P'$, it is a $x - y$ walk of shorter length than P that contains all of $W \cap V(P)$. Let $I(P'') = \bigcup_{u \in V(P'')} I(u)$. Then $[a^*, b^*] \subseteq I(P'')$. By the same argument as before, each vertex in L is adjacent to some vertex on P''. Thus, $V(P'') \cup L$ induces a connected subgraph of G that contains W and has fewer vertices than T', contradicting T' be a Steiner set of W. This contradiction completes our proof. $\square$

Finally, let us consider the subclass of strongly chordal graphs. A chordal graph is said to be *strongly chordal* if every cycle on six or more vertices contains a *strong chord*, i.e., a chord joining two vertices whose distance on the cycle is odd. Martin Farber proved in [4] the following forbidden induced subgraph characterization: a graph is strongly chordal if and only if it does not contain as an induced subgraph a cycle of length greater than three or a k-*sun* (i.e., a graph on $2k$ vertices consisting of a $2k$-cycle and a k-clique on the even vertices, see Figure 12d), for every $k \geq 3$. Observe that by this characterization both Ptolemaic graphs and interval graphs are strongly chordal. We have seen that for these two classes, the claim *Every Steiner set is geodetic* is true, it remains an interesting question whether the claim extends to the larger class of strongly chordal graphs (see Figure 13).

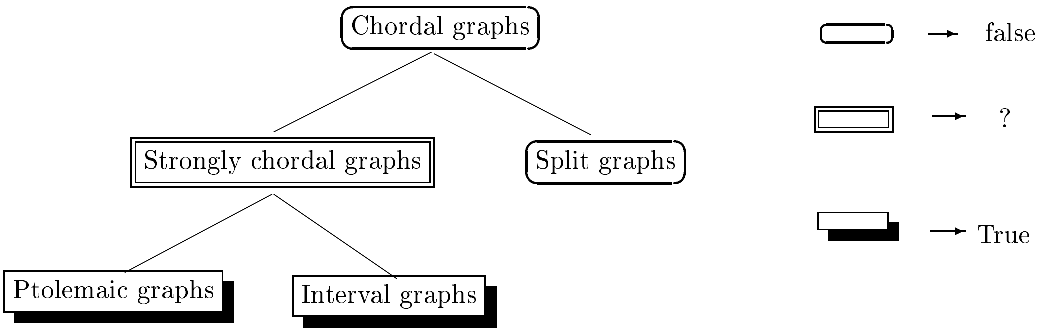

Figure 13: Is every Steiner set a geodetic set?

References

[1] G. Chartrand, F. Harary, P. Zhang, *On the hull number of a graph*, Ars Combinatoria, 57, 2000, pp. 129–138.

[2] G. Chartrand, P. Zhang, *The Steiner number of a graph*, Disc. Math., 242, 2002, pp. 41–54.

[3] F. F. Dragan, F. Nicolai, A. Brandstädt, *Convexity and HHD-free graphs*, SIAM J. Disc. Math., 12, 1999, pp. 119–135.

[4] M. Farber, *Characterizations of strongly chordal graphs*, Disc. Math., 43 (2-3), 1983, pp. 173–189.

[5] M. Farber, R. E. Jamison, *Convexity in graphs and hypergraphs*, SIAM J. Alg. Disc. Math., 7 (3), 1986, pp. 433–444.

[6] M. Farber, R. E. Jamison, *On local convexity in graphs*, Disc. Math., 66, 1987, pp. 231–247.

[7] S. Foldes, P. L. Hammer, *Split graphs*, Congressus Numerantium, XIX, 1977, pp. 311–315.

[8] F. Harary, E. Loukakis, C. Tsouros, *The geodetic number of a graph*, Math. Comput. Modelling, 17 (11), 1993, pp. 89–95.

[9] E. Howorka, *A characterization of distance-hereditary graphs*, Quart. J. Math. Oxford Ser. 2, 28 (112), 1977, pp. 417–420.

[10] E. Howorka, *A characterization of Ptolemaic graphs*, J. of Graph Theory, 5 (3), 1981, pp. 323–331.

[11] C. G. Lekkerkerker, J. Ch. Boland, *Representation of a finite graph by a set of intervals on the real line*, Fund. Math., 51, 1962/1963, pp. 45–64.

[12] I. M. Pelayo, *Not every Steiner set is geodetic, a note on "The Steiner number of a graph"*, Disc. Math., submmited.

[13] M. Van de Vel, *Theory of convex structures*, North-Holland, Amsterdam, MA, 1993.